Equivalence of Geometric $h < 1/2$ and Standard $c > 25$ Approaches to Two-Dimensional Quantum Gravity

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Abstract[1]

We show equivalence between the standard weak coupling regime $c > 25$ of the the two-dimensional quantum gravity and regime $h < 1/2$ of the original geometric approach of Polyakov [1, 2], developed in [3, 4, 5].

1 In this letter I shall demonstrate the equivalence of two approaches to the two-dimensional quantum gravity. The first approach, called geometric, goes back to Polyakov's original discovery [1], and was presented in [2]. According to [2], correlation functions of the Liouville vertex operators are represented by the functional integral with the Liouville action over all Riemannian metrics in a given conformal class with prescribed singularities at insertion points. This approach can be viewed as quantization of the hyperbolic geometry in two dimensions, and for the case of puncture operators, it has been extensively developed in our papers [3, 4, 5]. In these papers, we have computed conformal weights of puncture operators, the central charge, and also analyzed semi-classical behavior of the theory and its non-trivial relation with the Weil-Petersson geometry of Teichmüller spaces. Notably, the

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latter provides the Friedan-Shenker "program" of modular geometry [6] with a meaningful example. It is also worthwhile to mention that the first validation of geometric approach was provided by a rigorous proof [7] of the Polyakov's conjecture [2] that the classical Liouville action solves the problem of accessory parameters for the Fuchsian uniformization of Riemann surfaces, posed by Klein [8] and Poincaré [9].

Another approach to the Liouville theory, which we call standard, goes back to Braaten-Curtright-Thorn [11] who have computed conformal weights and the central charge of the theory using canonical quantization. Subsequently, these results were obtained by Knizhnik-Polyakov-Zamolodchikov [12], using methods of conformal field theory in the light-cone gauge, and by David-Distler-Kawai [13] in the conformal gauge. The connection between this approach and the hyperbolic geometry, uniformization and spectral theory of automorphic functions is less transparent than for the geometric one. On the other hand, standard approach allows one to use powerful methods of conformal field theory, such as free fields representation and analytic continuation, to study correlation functions of the theory.

Till now, these two approaches have looked look quite different and relation between them seemed obscure. In particular, geometric approach gives the following value to the central charge of the theory

$$c_{geom} = 1 + \frac{12}{h}, \tag{1}$$

with $h > 0$ being a bare coupling constant, whereas in standard approach

$$c_{stand} = 1 + 6(b + 1/b)^2, \tag{2}$$

where b is a corresponding Liouville coupling. Thus $c_{stand} > 25$, while c_{geom} takes all values greater than 1. Moreover, in geometric approach conformal weights of geometric vertex operators (i.e. that with charges corresponding to the Fuchsian uniformization) remain classical.

Recently, there was published a remarkable paper by A. and Al. Zamolodchikov's [14], where they provided an elegant geometric setting for standard approach and presented convincing arguments for Dorn-Otto conjecture [15] on structure constants for the operator algebra of the Liouville theory. Using formulation in [14], it is possible to show that methods developed in [4, 5] work for standard approach as well. In particular, as we shall demonstrate here, standard values for the conformal weights can be obtained by pure geometric

arguments, based on the exact form of the Liouville action. Formula (2) for the central charge can be also derived in the same manner as (1).

Moreover, these two apparently different approaches turn out to be equivalent in weak coupling regime $h < 1/2$ and $c > 25$! In order to understand this, we recall that dependence of physical parameters of the theory like conformal weights, central charge, etc., on the bare couplings is irrelevant and only the relation between them is fundamental. Therefore, one has a freedom of changing couplings, rescaling fields, etc., which in our case can be used as follows. First, start with the geometric approach and investigate whether it admits vertex operators with conformal weight 1. It turns out that for weak coupling regime $h < 1/2$ one has two such operators $V_{1,2}$ with charges

$$\alpha_{1,2}(h) = \frac{1 \mp \sqrt{1-2h}}{h}.$$

Second, introduce a new bare coupling constant

$$b(h) = \sqrt{\frac{h}{2}}\alpha_1(h) = \frac{1-\sqrt{1-2h}}{\sqrt{2h}},$$

so that $b(h) \to 0$ as $h \to 0$. As we shall show in the main text, formulas for conformal weights and central charges for geometric and standard approaches with corresponding coupling constants h and $b(h)$, become identical in weak coupling regime $h < 1/2$.

The paper is organized as follows. In the second section we briefly recall basic facts of geometric approach and present the geometric derivation of conformal weights. In the third section we summarize, in similar fashion, standard approach, and in the last section we elaborate the arguments for their equivalence in the weak coupling regime.

2 Here we outline main principles of geometric approach (see [3, 4, 5] for details). Let X be Riemann sphere $\mathbb{P}^1$ with n marked distinct points $z_1, \dots, z_{n-1}, z_n = \infty$. According to [2, 3, 4, 5], the n-point correlation function of Liouville vertex operators $V_\alpha(z) = e^{\alpha\phi(z,\bar{z})}$—spinless primary fields of the theory—is defined by the following functional integral

$$< V_{\alpha_1}(z_1)\cdots V_{\alpha_{n-1}}(z_{n-1})V_{\alpha_n}(\infty) >= \int_{\mathcal{C}(X)} \mathcal{D}\phi \; e^{-(1/2\pi h)S_X(\phi)}, \tag{3}$$

where h plays the role of Liouville coupling constant. Here "domain of integration" $\mathcal{C}(X)$ consists of all smooth conformal metrics $ds^2 = e^{\phi(z,\bar{z})}|dz|^2$ on X satisfying the following

asymptotics

$$\phi(z,\bar z)\simeq -\alpha_i h\log|z-z_i|^2,\ z\to z_i,\ i=1,\dots,n-1, \tag{4}$$

and

$$\phi(z,\bar z)\simeq (\alpha_n h-2)\log|z|^2,\ z\to z_n=\infty. \tag{5}$$

where $z=x+\sqrt{-1}y$ is a complex coordinate on $\mathbb{C}=\mathbb{P}^1\setminus\{\infty\}$. Charges α_i satisfy restriction $\alpha_i h<1$ (cf. [16]), which is equivalent to the finiteness of the area of X with respect to metric ds^2. In the limiting case $\alpha_i=1/h$, which corresponds to the punctures, asymptotics (4)–(5) should be modified

$$\phi(z,\bar z)\simeq -\log|z-z_i|^2-\log\log^2|z-z_i|,\ z\to z_i,\ i=1,\dots,n-1, \tag{6}$$

and

$$\phi(z,\bar z)\simeq -\log|z|^2-\log\log^2|z|,\ z\to z_n=\infty, \tag{7}$$

so that the total area of X remains finite. The action functional is given by

$$S_X(\phi)=\lim_{\epsilon\to 0}S_\epsilon(\phi),$$

where

$$\begin{aligned} S_\epsilon(\phi) &= \int\int_{X_\epsilon}(|\phi_z|^2+e^\phi)dx\wedge dy-\frac{h\sqrt{-1}}{2}\sum_{i=1}^{n-1}\alpha_i\int_{\gamma_i}\phi(\frac{d\bar z}{\bar z-\bar z_i}-\frac{dz}{z-z_i}) \\ &- \frac{\sqrt{-1}}{2}(\alpha_n h-2)\int_{\gamma_n}\phi(\frac{d\bar z}{\bar z}-\frac{dz}{z})-2\pi h^2\sum_{i=1}^{n-1}\alpha_i^2\log\epsilon-2\pi(\alpha_n h-2)^2\log\epsilon, \end{aligned} \tag{8}$$

where $X_\epsilon=X\setminus(\bigcup_{i=1}^{n-1}\{|z-z_i|<\epsilon\}\bigcup\{|z|>1/\epsilon\})$ and

$$\gamma_i(t)=z_i+\epsilon e^{2\pi\sqrt{-1}t},\ i=1,\dots,n-1,\ \gamma_n(t)=\frac{1}{\epsilon}e^{2\pi\sqrt{-1}t},\ 0\le t\le 1.$$

Note that in this form of the regularized action (cf. [7, 4, 14]) line integrals are necessary in order to ensure the proper asymptotic behavior (4)–(5). For the case $\alpha_i h=1$ one should modify the line integrals by adding terms $dz/((z-z_i)\log|z-z_i|)$ and their complex conjugates, as well as adding overall term $2\pi n\log|\log\epsilon|$ (cf. [7, 4]).

Conformal weights Δ_α of Liouville vertex operators $V_\alpha(z)$ are given by the following formula

$$\Delta_\alpha=\frac{h}{2}\alpha(\frac{2}{h}-\alpha), \tag{9}$$

which can be simply derived from the Liouville action [4, 5]. Namely, consider three-point correlation function $< V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(\infty) >$. Since, according to BPZ [17],

$$V_\alpha(\infty) = \lim_{z\to\infty} |z|^{4\Delta_\alpha} V_\alpha(z),$$

this correlation function has the form

$$< V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(\infty) >= \frac{C(\alpha_1,\alpha_2,\alpha_3)}{|z_1 - z_2|^{2\Delta_1+2\Delta_2-2\Delta_3}}, \tag{10}$$

where $\Delta_i = \Delta_{\alpha_i}$ and $C(\alpha_1,\alpha_2,\alpha_3)$ is the structure constant of the operator algebra of the theory. On the other hand, $< V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(\infty) >$ is represented by a functional integral (3) with $n = 3$. From the latter form it is easy to determine the coordinate dependence of the correlation function using global conformal invariance. Namely, consider fractional-linear transformation

$$\sigma z = \frac{z - z_1}{z_2 - z_1},$$

which maps Riemann surface X with three marked points z_1, z_2, ∞ onto normalized Riemann surface $\tilde{X}$ with three marked points $0, 1, \infty$. Since in geometric approach, e^ϕ transforms like $(1,1)$-tensor under local change of coordinates, one has

$$\tilde{\phi}(\sigma z) = \log|z_1 - z_2|^2 + \phi(z). \tag{11}$$

Now, straightforward computation yields

$$S_X(\phi) - S_{\tilde{X}}(\tilde{\phi}) = 2\pi h(\Delta_1 + \Delta_2 - \Delta_3)\log|z_1 - z_2|^2, \tag{12}$$

where $\Delta_\alpha = -h\alpha^2/2 + \alpha$. Note that the first term in this formula for Δ_α, which represents a "free-field contribution", comes from the transformation property of the surface and line integrals in the action (8). The second term, which equals to the classical conformal weight of $V_\alpha(z)$, comes entirely from the line integrals and the transformation law (11). Finally, observing that the local change of variables $\phi(z) \mapsto \phi(\sigma z)$ in the functional integral (3) leaves "integration measure" $\mathcal{D}\phi$ invariant, we get (10), where conformal weights are given by (9) and

$$C(\alpha_1,\alpha_2,\alpha_3) =< V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) > .$$

Note that conformal weights are invariant with respect to reflection $\alpha \mapsto 2/h - \alpha$ with the fixed point being $\alpha = 1/h$. For such α one can define puncture operator as

$$P(z) = \phi(z)V_{1/h}(z) = \frac{\partial}{\partial\alpha}V_\alpha(z)|_{\alpha=1/h},$$

(cf. [14, 16]) according with the double logarithm term in asymptotics (6)–(7).

It should be also noted that there is a discrete series of charges

$$\alpha_l = \frac{1}{h}(1 - \frac{1}{l}),\ l \text{ an integer} > 1 \text{ or } l = \infty,$$

which correspond to the Fuchsian uniformization of Riemann surface X with elliptic fixed points of finite order ($l < \infty$), or with punctures ($l = \infty$). For such α's, which correspond to geometric vertex operators, formula (9) gives the following conformal weights

$$\Delta_l = \frac{1}{2h}(1 - \frac{1}{l^2}).$$

It is remarkable that these weights coincide (times $1/h$) with the uniformization data, given by the coefficients at the second order poles of the Schwarzian derivative of the inverse function of the uniformization map of X [8, 9]. The latter quantity, according to [10], coincides with classical stress-energy tensor $T_{cl} = T(\phi_{cl})$, where

$$T(\phi) = \frac{1}{h}(\phi_{zz} - \frac{1}{2}\phi_z^2). \tag{13}$$

(See [3, 5] for more details and references).

Finally, the central charge of the Liouville theory can be computed from the short-distance behavior of the two-point correlation function of the stress-energy tensor in the presence of vertex operators

$$< T(z)T(w)V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(\infty) >= \int_{\mathcal{C}(X)} \mathcal{D}\phi\ T(\phi)(z)T(\phi)(w)\ e^{-(1/2\pi h)S_X(\phi)}. \tag{14}$$

Calculation from [4, 5] (performed for the case of puncture operators) gives the following expression for the central charge

$$c_{geom} = 1 + \frac{12}{h}, \tag{15}$$

where the only quantum correction to the semi-classical value $12/h$ comes from the one-loop contribution and is equal to 1.

We will not dwell here upon the relation of geometric approach to the Friedan-Shenker modular geometry and Weil-Petersson geometry of Teichmüller spaces, referring to [3, 4, 5].

3 Here we briefly recall, following [14], the basic facts from standard approach. The correlation function of vertex operators $e^{2a\phi}$ can be defined by the following functional integral

$$< e^{2a_1\phi(z_1)} \cdots e^{2a_n\phi(z_n)} >_Q = \int_{\mathcal{C}(X,Q)} \mathcal{D}\phi \; e^{-A_{X,Q}(\phi)}, \tag{16}$$

where "domain of integration" $\mathcal{C}(X,Q)$ now consists of all $(Q/2, Q/2)$-tensors $e^{\phi(z,\bar{z})}$ on X having the asymptotics (4) (with the replacement $n-1 \mapsto n$ and $\alpha \mapsto a$) and having the "charge Q" at infinity

$$\phi(z,\bar{z}) \simeq -Q \log |z|^2, \;\; z \to \infty. \tag{17}$$

The action is given by

$$A_{X,Q}(\phi) = \lim_{\epsilon \to 0} A_\epsilon(\phi),$$

where

$$\begin{aligned} A_\epsilon(\phi) \;\; &= \;\; \frac{1}{\pi} \int\int_{X_\epsilon} (|\phi_z|^2 + \pi\mu e^{2b\phi}) dx \wedge dy + \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^{n} a_i \int_{\gamma_i} \phi(\frac{d\bar{z}}{\bar{z} - \bar{z}_i} - \frac{dz}{z - z_i}) \\ &+ \;\; \frac{\sqrt{-1}Q}{2\pi} \int_{\gamma_\infty} \phi(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}) - 2(\sum_{i=1}^{n} a_i^2 + Q^2) \log \epsilon, \end{aligned} \tag{18}$$

with μ being a cosmological constant and b being a Liouville coupling constant.

Analyzing the three-point correlation function using the arguments from the previous section and using the new transformation law for e^ϕ, one gets the following expression for conformal weights Δ_a of vertex operators $e^{2a\phi(z)}$:

$$\Delta_a = a(Q - a). \tag{19}$$

The stress-energy tensor of the theory has the form

$$T_Q(\phi) = -\phi_z^2 + Q\phi_{zz}, \tag{20}$$

and the same arguments as in [4, 5] give the following formula for the central charge

$$c_{stand} = 1 + 6Q^2. \tag{21}$$

To relate global charge Q and Liouville coupling constant b, one imposes the condition that the "perturbation" operator $e^{2b\phi(z)}$ has conformal weight 1. It results in

$$Q = b + 1/b, \tag{22}$$

and gives standard restriction $c > 25$ on the central charge of the theory in the weak coupling regime (which corresponds to real b). For further discussion of this approach, and correlation functions properties in particular, we refer to [14, 15] and references therein.

Arguments presented here are nothing but geometric interpretation of KPZ-DDK results. Note that accurate (cf. [18, Sect. 3.7]) definition of the Liouville action requires line integrals terms, which play fundamental role in geometric derivation of conformal weights. Being linear in the Liouville field, these terms do not contribute to the perturbation theory.

4. Here we compare both approaches and establish their equivalence for weak coupling regime $h < 1/2$ and $c > 25$.

At first glance, these approaches look strikingly different. Indeed, one has $c_{stand} > 25$, whereas $c_{geom} > 1$. What is more, in standard approach, operator $e^{b\phi(z)}$ has conformal weight 1, whereas in geometric approach conformal weight of $V_1(z) = e^{\phi(z)}$ is $1 - h/2$. The latter even looks like a drawback of the geometric approach. However, that is not really so, since in standard approach, classical condition of e^{ϕ} being a $(Q/2, Q/2)$-tensor is also "violated" after quantization: $\Delta_{1/2} = Q/2 - 1/4 \neq Q/2$. Nevertheless, there is nothing wrong with these results, since in the formalism of functional integration, one should integrate over all classical configurations, whether they are conformal Riemannian metrics in geometric approach, or $(Q/2, Q/2)$-tensors in standard approach.

Still, the importance of the latter remark is in the fundamental role played by vertex operators of conformal weight 1. It turns out that in the weak coupling regime of geometric approach, it is also possible to find such operators. Simply solving equation $\Delta_\alpha = 1$, we get the roots

$$\alpha_{1,2} = \frac{1 \mp \sqrt{1 - 2h}}{h},$$

which are real if and only if $h < 1/2$. In this regime, $c_{geom} > 25$, which suggests the equivalence with standard approach.

Indeed, starting from geometric approach and setting

$$Q(h)=\sqrt{\frac{2}{h}},\ \alpha=Qa$$

(thus effectively rescaling Liouville field ϕ) we get from (9) and (15) formulas (19) and (21):

$$\Delta_a=\Delta_\alpha=a(Q-a),\quad c_{stand}=1+6Q^2.$$

Moreover, real roots $\alpha_{1,2}$ satisfy $\alpha_1\alpha_2=Q^2,\ \alpha_1+\alpha_2=Q^2$, so that $a_1a_2=1,\ a_1+a_2=Q$. Thus, introducing

$$b(h)=a_1=\frac{1-\sqrt{1-2h}}{\sqrt{2h}},$$

which is real for $h<1/2$, we get the constraint (22):

$$Q(h)=b(h)+1/b(h),$$

which establishes the equivalence between geometric and standard approaches.

It is well known that it is extremely difficult to extend standard approach to strongly coupled regime $c<25$ (formally coupling constant b in (2) becomes pure imaginary). On the other hand, geometric approach seems to be well-defined for all positive values of h, and in particular, for strongly coupled regime $h\geq 1/2$, for which $c_{geom}\leq 25$. It suggests that this regime of geometric approach can be considered as strongly coupled regime for standard approach. Characteristic novel feature of this regime is that the theory no longer contains operators with conformal weight 1. It is worthwhile to further investigate this interesting possibility.

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